

# $\mathbb{Q}_\ell$ -COHOMOLOGY PROJECTIVE PLANES AND SINGULAR ENRIQUES SURFACES IN CHARACTERISTIC TWO

MATTHIAS SCHÜTT

ABSTRACT. We classify singular Enriques surfaces in characteristic two supporting a rank nine configuration of smooth rational curves. They come in one-dimensional families defined over the prime field, paralleling the situation in other characteristics, but featuring novel aspects. Contracting the given rational curves, one can derive algebraic surfaces with isolated ADE-singularities and trivial canonical bundle whose  $\mathbb{Q}_\ell$ -cohomology equals that of a projective plane.

## 1. INTRODUCTION

The study of Enriques surfaces forms one of the centerpieces for the Enriques–Kodaira classification of algebraic surfaces. The subtleties and special properties which they already display over the complex numbers are further augmented in characteristic two where additional intriguing features arise. Despite great recent progress (see [12]), there are still many open problems, both on the abstract and the explicit side. This paper contributes to both sides by considering Enriques surfaces which carry a rank nine configuration of smooth rational curves. Contracting the curves, we obtain a remarkable object: a singular normal surface with the same étale cohomology as projective space  $\mathbb{P}^2$ .

Enriques surfaces admitting such configurations of smooth rational curves have previously been studied successfully in [5] and [17] over  $\mathbb{C}$  and in [18] over fields of odd characteristic. In characteristic two, however, the theory of Enriques surfaces features many subtleties and surprises which will also play a lead role in this paper. Hence it is quite remarkable that some of the techniques from [18] can be adapted for one big class of Enriques surfaces in characteristic two, namely the so-called singular Enriques surfaces whose Picard scheme equals the group scheme  $\mu_2$ . Our results parallel those from odd characteristic in quality although they are quite different in quantity.

---

*Date:* March 28, 2017.

2010 *Mathematics Subject Classification.* 14J28; 14J27.

Partial funding by ERC StG 279723 (SURFARI) is gratefully acknowledged.

**Theorem 1.1.** *There are exactly 20 configurations  $R$  of smooth rational curves of rank 9 realized on singular Enriques surfaces in characteristic two:*

$$\begin{aligned} A_9, A_8 + A_1, A_7 + A_2, A_7 + 2A_1, A_6 + A_2 + A_1, A_5 + A_4, A_5 + A_3 + A_1, \\ A_5 + 2A_2, A_5 + A_2 + 2A_1, 2A_4 + A_1, 3A_3, A_3 + 3A_2, D_9, D_8 + A_1, \\ D_6 + A_3, D_5 + A_4, E_8 + A_1, E_7 + A_2, E_6 + A_3, E_6 + A_2 + A_1. \end{aligned}$$

*For each the following hold:*

- (i) *the root types are supported on 1-dimensional families of Enriques surfaces;*
- (ii) *the moduli spaces are irreducible except for  $R = A_8 + A_1$  and  $A_5 + 2A_2$  both of which admit two different moduli components;*
- (iii) *each family has rational base and is defined over  $\mathbb{F}_2$ ;*
- (iv) *each family can be parametrized explicitly, see Summary 6.2 and Table 3.*

The paper is organized as follows. After reviewing some relevant theory around elliptic fibrations, we will right away jump at ruling out many root types to occur on singular Enriques surfaces (a good portion of them existing in characteristic zero in fact). The existence and uniqueness part of Theorem 1.1 builds on a rather explicit base change technique which abstractly goes back to Kondō over  $\mathbb{C}$  ([8]) and is extended to characteristic two in Sections 5, 6. By way of this technique, all Enriques surfaces and root types from Theorem 1.1 fall into two kinds of families which we work out explicitly in the concluding sections. Throughout, we will illustrate the methods and ideas involved with a series of instructive examples related to the root type  $R = A_5 + 2A_2$ .

**Convention 1.2.** All root lattices of type  $A_n, D_k, E_l$  are assumed to be negative-definite.

## 2. ELLIPTIC FIBRATIONS

Regardless of the characteristic, an Enriques surface can be defined to be a smooth minimal algebraic surface  $S$  such that

$$K_S \equiv 0, \quad b_2(S) = 10$$

(where the Enriques–Kodaira classification, extended by Bombieri and Mumford in [1], [2], [15], enters implicitly). Here  $\equiv$  indicates numerical equivalence and  $b_i(S)$  denotes the  $i$ th Betti number of  $S$  for the  $\ell$ -adic étale cohomology (with some auxiliary prime  $\ell \neq p$ ). Moreover, one has

$$\chi(\mathcal{O}_S) = 1, \quad b_1(S) = 0.$$

From now on we assume that we work over an algebraically closed field  $K$  of characteristic  $p = 2$ . Then Enriques surfaces fall into three cases, depending on their Picard scheme  $\text{Pic}^\tau(S)$  which runs through all group schemes of length two

(or, equivalently, depending on the action of the absolute Frobenius morphism on  $H^1(S, \mathcal{O}_S)$  which can be zero- or one-dimensional):

$$\begin{array}{ll} \text{classical:} & \text{Pic}^\tau(S) = \mathbb{Z}/2\mathbb{Z} \\ \text{singular:} & \text{Pic}^\tau(S) = \mu_2 \\ \text{supersingular:} & \text{Pic}^\tau(S) = \alpha_2 \end{array}$$

All three cases are unified by the way in which they are governed by genus one fibrations, i.e. morphisms

$$(1) \quad f : S \rightarrow \mathbb{P}^1$$

whose general fiber are curves of arithmetic genus one. Necessarily, there are multiple fibers involved (of multiplicity two), but their configuration is genuinely different from the picture in all other characteristics:

classical	two multiple fibers, both smooth ordinary or of additive type
singular	one multiple fiber, smooth ordinary or of multiplicative type
supersingular	one multiple fiber, smooth supersingular or of additive type

Abstractly, the fibration is encoded in the lattice

$$\text{Num}(S) = \text{Pic}(S)/ \equiv \cong U + E_8$$

where  $U$  denote the hyperbolic plane (cf. [6]). Precisely, the genus one fibration (1) gives rise to a primitive isotropic effective vector

$$0 < E \in \text{Num}(S)$$

by way of the support of a multiple fiber; by construction, the linear system  $|2E|$  has no base points and thus coincides with (1). Conversely, given any non-trivial isotropic vector  $E \in \text{Num}(S)$ , it

- is effective or anti-effective by Riemann–Roch (so say  $E > 0$ ),
- can be made primitive by dividing by some appropriate constant, and
- one can eliminate the base locus of the linear system  $|2E|$  by successive reflections in the classes of smooth rational curves.

Recall that the resulting effective isotropic divisor  $E'$  is often called a half-pencil.

We emphasize that smooth rational curves are a rather delicate matter on Enriques surfaces since, unlike on K3 surfaces,  $(-2)$ -vectors in  $\text{Num}(S)$  need not be effective neither anti-effective. Nonetheless, the above approach will be the key to our findings since we can set it up in such a way that we retain enough control over the smooth rational curves provided by the rank 9 configuration (as explored in [17], [18]).

Instead of going into the details of the gluing techniques and discriminant forms from [17], [18], we illustrate our methods and ideas with the following instructive example which will keep on reappearing throughout this paper.

*Example 2.1* ( $R = A_5 + 2A_2$ ). Assume that the Enriques surface  $S$  contains a configuration of smooth rational curves of type  $R = A_5 + 2A_2$ . Up to isometry,  $R$  admits a unique embedding into  $\text{Num}(S)$  with primitive closure

$$R' = (R \otimes \mathbb{Q}) \cap \text{Num}(S) = E_7 + A_2$$

and orthogonal complement

$$R^\perp = \mathbb{Z}H \subset \text{Num}(S), \quad H^2 = 6.$$

Using dual vectors in standard notation (cf. [17]), there are several ways to exhibit isotropic vectors  $E \in \text{Num}(S)$ . We highlight two of them:

- (1)  $a_1^\vee \in A_2^\vee \implies E = a_1^\vee + H/3, \quad E^\perp \cap R = A_5 + A_2 + A_1.$
- (2)  $e_7^\vee \in E_7^\vee \subset A_5^\vee \implies E = e_7^\vee + H/2, \quad E^\perp \cap R = 4A_2.$

Let us point out two properties which will become important momentarily: in either case,  $E$  is primitive (since  $E.a_1 = 1$  resp.  $E.e_7 = 1$  by construction), while leaving a large root lattice inside  $R$  perpendicular. Up to changing sign and subtracting the base locus,  $|2E|$  will thus induce a genus one fibration on  $S$ .

### 3. JACOBIAN VS EXTREMAL ELLIPTIC SURFACES

It is exactly the approach of Example 2.1 which we will now pursue in order to get an idea of the Jacobian of  $|2E|$ . To this end, we restrict to **singular Enriques surfaces** since then (1) always defines an elliptic fibration in the sense that the general fiber is a smooth curve of genus one (as opposed to quasi-elliptic fibration). Hence the Jacobian  $\text{Jac}(f)$  is a rational elliptic surface. We claim that we can always arrange for  $\text{Jac}(f)$  to be rather special, namely to have finite Mordell–Weil group. Equivalently, by the Shioda–Tate formula, the fibers support a root lattice of rank 8. Following [14], such rational elliptic surfaces are called *extremal*.

**Proposition 3.1.** *Let  $S$  be a singular Enriques surface admitting a configuration  $R$  of smooth rational curves of rank 9. Then  $S$  has an elliptic fibration (1) whose Jacobian is an extremal rational elliptic surface.*

*Proof.* The proof is the same as in [18]. In short, there are 38 root types  $R$  of rank 9 embedding into  $U + E_8$ . For each of them, we exhibit a suitable isotropic vector  $E$  as in Example 2.1 such that  $E^\perp \cap R$  is a root lattice  $R_0$  of rank 8 (see Table 1).  $R_0$  is not affected by a potential sign change in  $E$ , but it may be moved around by the reflections needed to eliminate the base locus of  $|2E|$ . In the end, however,  $R_0$  will still be supported on  $(-2)$ -curves, and at the same time on the fibers of  $|2E|$  which suffices to conclude the proposition.  $\square$

In the next section, we will rule out almost half of the above 38 root types (the lower 18 root types from Table 1 such that the 20 root types from Theorem 1.1 remain). For this purpose, it will be extremely useful to have the classification of extremal rational elliptic surfaces in characteristic two handy (due to W. Lang

$R$	$R'$	$H^2$	$E$	$R_0$
$A_9$	$A_9$	10	$a_2^\vee + 2h/5$	$A_7 + A_1$
$A_8 + A_1$	$A_8 + A_1$	18	$(a_3^\vee, 0) + h/3$	$A_5 + A_2 + A_1$
	$E_8 + A_1$	2	$(a_3^\vee, 0) + h$	$A_5 + A_2 + A_1$
$A_7 + A_2$	$E_7 + A_2$	6	$(a_2^\vee, 0) + h/2$	$A_5 + A_2 + A_1$
$A_7 + 2A_1$	$E_8 + A_1$	2	$(a_4^\vee, 0, 0) + h$	$2A_3 + 2A_1$
$A_6 + A_2 + A_1$	$A_6 + A_2 + A_1$	42	$(a_1^\vee, 0, 0) + h/7$	$A_5 + A_2 + A_1$
$A_5 + A_4$	$A_5 + A_4$	30	$(0, a_2^\vee) + h/5$	$A_5 + A_2 + A_1$
$A_5 + A_3 + A_1$	$E_6 + A_3$	24	$(a_2^\vee, 0, 0) + h/3$	$2A_3 + 2A_1$
$A_5 + 2A_2$	$E_7 + A_2$	6	$(0, 0, a_1^\vee) + h/3$	$A_5 + A_2 + A_1$
$A_5 + A_2 + 2A_1$	$E_8 + A_1$	2	$(0, 0, 0, a_1^\vee) + h/2$	$A_5 + A_2 + A_1$
$2A_4 + A_1$	$E_8 + A_1$	2	$(0, 0, a_1^\vee) + h/2$	$2A_4$
$3A_3$	$D_9$	4	$(a_2^\vee, 0, 0) + h/2$	$2A_3 + 2A_1$
$A_3 + 3A_2$	$A_3 + E_6$	12	$(a_1^\vee, 0, 0, 0) + h/4$	$4A_2$
$D_9$	$D_9$	4	$d_9^\vee + 3h/4$	$A_8$
$D_8 + A_1$	$E_8 + A_1$	2	$d_8^\vee + h$	$A_7 + A_1$
$D_6 + A_3$	$D_9$	4	$(0, a_2^\vee) + h/2$	$D_6 + 2A_1$
$D_5 + A_4$	$D_5 + A_4$	20	$(d_5^\vee, 0) + h/4$	$2A_4$
$E_8 + A_1$	$E_8 + A_1$	2	$(e_8^\vee, 0) + h$	$E_7 + A_1$
$E_7 + A_2$	$E_7 + A_2$	6	$(e_3^\vee, 0) + h$	$A_5 + A_2 + A_1$
$E_6 + A_3$	$E_6 + A_3$	12	$(e_6^\vee, 0) + h/3$	$D_5 + A_3$
$E_6 + A_2 + A_1$	$E_8 + A_1$	2	$(e_1^\vee, 0, 0) + h$	$A_5 + A_2 + A_1$
$E_7 + 2A_1$	$E_8 + A_1$	2	$(e_2^\vee, 0, 0) + h$	$D_6 + 2A_1$
$D_7 + 2A_1$	$D_9$	4	$(d_1^\vee, 0, 0) + h/2$	$D_6 + 2A_1$
$D_6 + A_2 + A_1$	$E_7 + A_2$	6	$(0, a_1^\vee, 0) + h/3$	$D_6 + 2A_1$
$D_6 + 3A_1$	$E_8 + A_1$	2	$(d_2^\vee, 0, 0, 0) + h$	$D_4 + 4A_1$
$D_5 + A_3 + A_1$	$E_8 + A_1$	2	$(d_2^\vee, 0, 0) + h$	$2A_3 + 2A_1$
$D_5 + D_4$	$D_9$	4	$(d_1^\vee, 0) + h/2$	$2D_4$
$D_5 + 4A_1$	$D_9$	4	$d_1^\vee + h/2$	$D_4 + 4A_1$
$D_4 + A_3 + 2A_1$	$D_9$	4	$(d_1^\vee, 0, 0, 0) + h/2$	$2A_3 + 2A_1$
$2D_4 + A_1$	$E_8 + A_1$	2	$(0, 0, a_1^\vee) + h/2$	$2D_4$
$D_4 + 5A_1$	$E_8 + A_1$	2	$a_1^\vee + h/2$	$D_4 + 4A_1$
$D_4 + A_2 + 3A_1$	$E_7 + A_2$	6	$a_2^\vee + h/3$	$D_4 + 4A_1$
$A_4 + A_3 + 2A_1$	$A_4 + D_5$	20	$(a_1^\vee, 0, 0, 0) + h/5$	$2A_3 + 2A_1$
$2A_3 + A_2 + A_1$	$E_7 + A_2$	6	$(0, 0, a_1^\vee, 0) + h/3$	$2A_3 + 2A_1$
$2A_3 + 3A_1$	$E_8 + A_1$	2	$(0, 0, 0, 0, a_1^\vee) + h/2$	$2A_3 + 2A_1$
$A_3 + 6A_1$	$D_9$	4	$a_2^\vee + h/2$	$8A_1$
$4A_2 + A_1$	$E_8 + A_1$	2	$(0, 0, 0, 0, a_1^\vee) + h/2$	$4A_2$
$A_2 + 7A_1$	$A_2 + E_7$	6	$a_2^\vee + h/3$	$8A_1$
$9A_1$	$E_8 + A_1$	2	$a_1^\vee + h/2$	$8A_1$

TABLE 1. Isotropic vectors and root lattices (Proposition 3.1)

in [9], [10]). Before going into the details, we note the following important observation:

**Lemma 3.2.** *There are no extremal rational elliptic surfaces such that the fiber components off the zero section generate the root lattice*

$$2A_3 + 2A_1, \quad D_6 + 2A_1 \quad \text{or} \quad 2D_4.$$

*Proof.* Of course, this follows from Lang's classification, but for the sake of completeness (and since it fits nicely into the scheme of our arguments), we give a quick proof. Let  $X$  be a rational elliptic surface and denote by  $R$  the root lattice generated by the fiber components perpendicular to the zero section. The theory of Mordell–Weil lattices [20] provides an equality

$$\text{MW}(X) = E_8/R.$$

In the above cases, this would imply

$$(\mathbb{Z}/2\mathbb{Z})^2 \subset \text{MW}(X)$$

which is clearly impossible in characteristic two.  $\square$

The following table lists all the extremal rational elliptic surfaces in characteristic two relevant to our issues. We follow the notation from [14], even though the fiber types often degenerate compared to  $\mathbb{C}$ .

notation	Weierstrass eqn.	sing. fibers	MW
$X_{9111}$	$y^2 + xy + t^3y = x^3$	$I_9/0, I_1/t^3 + 1$	$\mathbb{Z}/3\mathbb{Z}$
$X_{8211}$	$y^2 + xy + t^2y = x^3 + t^2x^2$	$I_8/0, III/\infty$	$\mathbb{Z}/4\mathbb{Z}$
$X_{6321}$	$y^2 + xy + t^2y = x^3$	$I_6/0, IV/\infty, I_2/1$	$\mathbb{Z}/6\mathbb{Z}$
$X_{5511}$	$y^2 + (t+1)xy + t^2y = x^3 + tx^2$	$I_5/0, \infty, I_1/t^2 + t + 1$	$\mathbb{Z}/5\mathbb{Z}$
$X_{3333}$	$y^2 + xy + (t^3+1)y = x^3$	$I_3/0, t^3 + 1$	$(\mathbb{Z}/3\mathbb{Z})^2$
$X_{431}$	$y^2 + txy + t^2y = x^3$	$IV^*/0, I_3/\infty, I_1/1$	$\mathbb{Z}/3\mathbb{Z}$
$X_{321}$	$y^2 + xy = x^3 + tx$	$III^*/\infty, I_2/0$	$\mathbb{Z}/2\mathbb{Z}$
$X_{141}$	$y^2 + xy = x^3 + t^2x$	$I_4/0, I_1^*/\infty$	$\mathbb{Z}/4\mathbb{Z}$

TABLE 2. Extremal rational elliptic surfaces in characteristic two

*Remark 3.3.* There are a few further extremal rational elliptic surfaces in characteristic two, but they will not be relevant to our issues.

#### 4. NON-EXISTENCE OF CERTAIN ROOT TYPES

In order to rule out the lower 18 root types from Table 1, we have to draw a few more consequences from the other given data. Recall that the primitive isotropic vector  $E$  is effective, possibly after changing sign while subtracting the base locus amounts to a succession  $\sigma_0$  of reflections in smooth rational curves; here  $|2\sigma_0(E)|$  induces the elliptic fibration (1). In each case, there is a single  $(-2)$ -curve  $C$  in  $R$  whose dual vector is involved in forming  $E$ . Hence  $\sigma_0(C) \cdot \sigma_0(E) = 1$ , and it follows that  $\sigma_0(C)$  comprises a smooth rational

bisection  $B$  of the fibration  $|2\sigma_0(E)|$  plus possibly some smooth rational curves  $C_i$  contained in the fibers of (1); again these can be eliminated by reflections, and since naturally  $C_i \cdot \sigma_0(E) = 0$ , these reflections do not affect  $\sigma_0(E)$ .

**Summary 4.1.** There is a composition of reflections  $\sigma$  such that  $\sigma(E)$  is a half-pencil and  $B = \sigma(C)$  is a smooth rational bisection while  $\sigma(R_0)$  is supported on the fibers and meets  $B$  in a prescribed way.

We shall now take a closer look at  $\sigma(R_0)$ . This root lattice consists of  $(-2)$ -divisors, each effective or anti-effective and supported on  $(-2)$ -curves (the fiber components). Note that we cannot claim in general that  $\sigma(R_0)$  consists of  $(-2)$ -curves, and even if that were to hold true, it would not imply that the orthogonal summands  $R_v$  of  $R_0$  determine the singular fibers of  $|2\sigma(E)|$  as extended Dynkin diagrams  $\tilde{R}_v$ . Yet we can utilize the bisection  $B$  very much to our advantage.

**Lemma 4.2.** *Assume that  $C$  meets less than two of the curves forming  $R_0 = \oplus_v R_v$  (where all orthogonal summands  $R_v$  are irreducible root lattices). Then the only possible fibre configuration for  $|2\sigma(E)|$  comprises  $\tilde{R}_v$  fibers.*

*Proof.* This is Criterion 7.1 from [18]. The proof carries over literally as it does not depend on the characteristic at all.  $\square$

**Proposition 4.3.** *The 18 lower root lattices in Table 1 cannot be supported on singular Enriques surfaces in characteristic two.*

*Proof.* We apply Lemma 4.2 to the data from Table 1. Except for  $R = A_3 + 6A_1$ , it follows that  $R_0$  determines the singular fibers of  $|2\sigma(E)|$ . But then, except for  $R = 4A_2 + A_1$ , neither case is compatible with Lang's classification (see especially Lemma 3.2). Meanwhile one could try to rule out  $R = 4A_2 + A_1$  using a 3-length argument on the K3-cover based on lifting to characteristic zero following [3] but this would be more involved than in the proof of [18, Prop. 5.4] (depending, for instance, on [11, Prop. 4.1]). Hence we postpone this case for a direct explicit analysis in 8.2.

It remains to exclude the root type  $R = A_3 + 6A_1$  (which over  $\mathbb{C}$  is not compatible with the orbifold Bogomolov–Miyaoka–Yau inequality). The two  $(-2)$ -curves  $a_1, a_3 \subset A_3$  which meet  $C = a_2$  are taken to  $(-2)$ -divisors  $\sigma(a_1), \sigma(a_3)$  whose support could potentially involve two different simple components of a single fiber  $F_0$  both of which are met by the bisection  $B = \sigma(a_2)$ . All other summands  $A_1$ 's from  $R$  are either mapped into  $F_0$ , or they give rise to fibers of type  $\tilde{A}_1$  (which follows as in the proof of Lemma 4.2). A priori, this leads to the configurations

$$\tilde{D}_4 + 4\tilde{A}_1, \quad 2\tilde{D}_4, \quad \tilde{D}_6 + 2\tilde{A}_1, \quad \tilde{E}_7 + \tilde{A}_1$$

(no  $\tilde{E}_8$  since it only has one simple fiber component), but by inspection of Lang's classification (and Lemma 3.2), this only leaves  $\tilde{E}_7 + \tilde{A}_1$ . By some reflections in fiber components, we can arrange for  $a_1, a_3$  to map to the two simple fiber components met by  $B$ . But then the 5 orthogonal copies of  $A_1$  embed into the

fiber minus these two components, and by orthogonality, minus the adjacent components, i.e. into  $D_4$  which is impossible for rank reasons.  $\square$

We illustrate these ideas by continuing to investigate the root type  $R = A_5 + 2A_2$  as initiated in Example 2.1.

*Example 4.4* ( $R = A_5 + 2A_2$  cont'd). Consider the root type  $R = A_5 + 2A_2$  with the second isotropic vector from Example 2.1. The fiber configurations accommodating  $R_0 = 4A_2$  are

$$4\tilde{A}_2, \quad \tilde{E}_6 + \tilde{A}_2, \quad \tilde{E}_8.$$

However, the last configuration is not compatible with the bisection meeting some fiber component with multiplicity one, but the other two work perfectly fine.

For completeness, we record the following nice consequence of Proposition 4.3:

**Corollary 4.5.** *The orbifold Bogomolow–Miyaoka–Yau inequality holds for any  $\mathbb{Q}_\ell$ -cohomology projective plane obtained from a singular Enriques surface in characteristic two.*

## 5. BASE CHANGE CONSTRUCTION

To translate the data from Table 1 into geometric information, and eventually into explicit equations, we elaborate on a base change construction which was originally due to Kondō in [8] and later extended in [4]. Notably, a singular Enriques surface  $S$  in characteristic two still has a universal K3 cover

$$X \rightarrow S$$

which exhibits  $S$  as a quotient of  $X$  by a fixed point free involution  $\tau$  (unlike for classical and supersingular Enriques surfaces where  $X$  may only be K3-like etc). Note that  $X$  inherits an elliptic fibration from  $S$  (for instance by pulling back the half-pencil  $\sigma(E)$ ),

$$(2) \quad X \rightarrow \mathbb{P}^1,$$

which presently is endowed with the two disjoint sections  $O, P$  which the smooth rational bisection  $B$  splits into. The next result translates Kondō's work [8] over  $\mathbb{C}$  to characteristic two, combining the Enriques involution  $\tau$  with translation by  $P$ , denoted by  $t_P$ . It has been obtained independently by Martin in [13] in relation with Enriques surfaces with finite automorphism groups.

**Proposition 5.1.** *There is an involution  $\iota$  on  $X$  such that*

$$\tau = t_P \circ \iota.$$

*Proof.* The proof follows the line of arguments in [17] although we have to make crucial modifications in order to account for the special features in characteristic two. Consider the automorphism

$$\iota = t_P^{-1} \circ \tau \in \text{Aut}(X).$$



Then  $\iota$  fixes  $O$  as a set, with a single fixed point in the ramified fiber, say  $F_\infty$ . We now turn to  $\iota^2$  which the proposition states to be the identity. Since  $\iota^2$  fixes fibers as sets and  $O$  pointwise, it acts as an automorphism on the generic fiber  $X_\eta$  of (2). In particular,

$$\text{ord}(\iota^2) \in \{1, 2, 3, 4, 6\}.$$

If  $3 \mid \text{ord}(\iota^2)$ , then the induced action of  $\iota^2$  on the regular 1-form of  $X_\eta$  involves a primitive cube root of unity. Naturally, this extends to the action on a regular 2-form  $\omega$  on  $X$ . On the other hand, the involution  $\tau^*$  leaves  $\omega$  invariant, and so does  $(t_P)^*$ , so we obtain a contradiction.

It remains to rule out the cases  $\text{ord}(\iota^2) \in \{2, 4\}$ . We switch to the ramified fiber  $F_\infty$ . If  $F_\infty$  is smooth, then  $\iota \in \text{Aut}(F_\infty)$  (since  $\iota$  fixes  $O|_{F_\infty}$ ), and it follows right away that

$$\text{ord}(\iota) = 4, \quad \text{ord}(\iota^2) = 2 \quad \text{and} \quad F_\infty \text{ is supersingular with } j = 0.$$

Here  $F_\infty$  is isomorphic to the curve with Weierstrass equation  $y^2 + y = x^3$ , so the ramified fiber has only one two-torsion point:  $F_\infty[2] = \{O|_{F_\infty}\}$ . In comparison, the definition of  $\iota$  shows

$$\iota(P) = -P, \quad \text{so} \quad \iota(P|_{F_\infty}) = -P|_{F_\infty} \quad \text{and in particular} \quad \iota^2(P|_{F_\infty}) = P|_{F_\infty}.$$

But then we have

$$F_\infty[2] = \ker(\iota^2|_{F_\infty}) = \{O|_{F_\infty}, P|_{F_\infty}\},$$

yielding the required contradiction.

If  $F_\infty$  is singular, i.e. of Kodaira type  $I_{2n}$  for some  $n \geq 1$ , then  $\iota$  acts as an automorphism of the smooth locus  $\Theta_0^\#$  of the identity component  $\Theta_0$  of  $F_\infty$ . But this leads to the multiplicative group:

$$(3) \quad \Theta_0^\# \cong \mathbb{G}_m \implies \text{Aut}(\Theta_0^\#) = \mathbb{Z}/2\mathbb{Z} \implies (\iota^2)|_{\Theta_0^\#} = \text{id}.$$

However, with  $\iota^2$  acting as inversion on the generic fiber, it cannot act as identity on any component of a multiplicative fiber (for instance, inversion interchanges the nodes where the fiber components intersect). This contradiction completes the proof of Proposition 5.1.  $\square$

**Corollary 5.2.** *The involution  $\iota$  leads us back to the rational elliptic surface  $\text{Jac}(f)$ :*

$$X/\iota = \text{Jac}(f).$$

*Proof.* As the involution interchanges the unramified fibers and fixes  $O$  as a set, the quotient  $X/\iota$  inherits an elliptic fibration with section induced from  $O$ . It remains to study the ramified fiber  $F_\infty$ .

If  $F_\infty$  is smooth, then the proof of Proposition 5.1 shows that  $\iota$  restricts to an automorphism of  $F_\infty$ , with the following two possibilities:

$$\iota|_{F_\infty} = \pm \text{id}.$$

In fact, it is easy to see that  $\iota|_{F_\infty} = \text{id}$ , since otherwise  $\tau$  would fix any point  $Q_\infty \in F_\infty$  with  $2Q_\infty = P|_{F_\infty}$ . But then the Euler-Poincaré characteristic

reveals that  $X/\iota$  is a rational elliptic surface, and since it has exactly the same fibers as the Enriques surface  $Y$  (including the half-pencil equalling  $F_\infty$ ), we deduce that  $X/\iota = \text{Jac}(f)$  as stated.

If  $F_\infty$  is non-smooth, i.e. multiplicative of type  $I_{2n}$  ( $n > 0$ ), then we number the components cyclically as usual, starting from the zero component  $\Theta_0$  met by  $O$  up to  $\Theta_{2n-1}$ . Again we have seen in the proof of Proposition 5.1 that  $\iota$  restricts to an automorphism of the smooth locus  $\Theta_0^\#$  of  $\Theta_0$ ; more precisely, by (3),  $\iota|_{\Theta_0^\#}$  either is the identity or inversion in  $\Theta_0^\# \cong \mathbb{G}_m$ . Note that for  $\tau$  to be fixed point free,  $P$  intersects a different component than  $\Theta_0$  so that  $t_P$  induces a rotation of the fiber components. Anyway, if

$$\iota|_{\Theta_0^\#} \neq \text{id},$$

then one checks that  $\tau$  either fixes a component (as a set, but then it contains a fixed point), or it interchanges two adjacent components (such that the intersection point is fixed). Hence we infer that

$$\iota|_{\Theta_0^\#} = \text{id},$$

so that all fiber components are fixed by  $\iota$  as sets, and  $P$  has to meet  $\Theta_n$  for  $\tau$  to have order two. Subsequently, this implies that  $X/\iota$  attains a fiber of type  $I_n$ , and we conclude as before.  $\square$

The corollary explains the title of this section:  $X$  is a quadratic base change of the Jacobian of the elliptic fibration (1) on the Enriques surface  $S$ . It is instructive to distinguish whether  $P$  is two-torsion or not, since in the first case, it will descend to  $\text{Jac}(f)$  (as it is invariant under the action of  $\iota$ ) while in the second case it is honestly anti-invariant for the action of  $\iota$  by definition.

*Example 5.3* ( $R = A_5 + 2A_2$  cont'd). We illustrate the two cases above with possible configurations supporting the root type  $R = A_5 + 2A_2$ . Arguing with the first isotropic vector from Example 2.1, Lemma 4.2 implies that  $\text{Jac}(f) = X_{6321}$ . For the 2-torsion case, we find the following configuration of  $(-2)$ -curves with ramified  $I_6$  fiber:

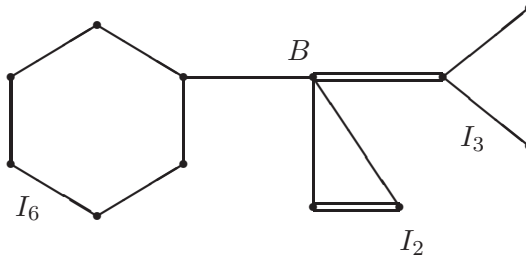


FIGURE 1. Configuration with multiple  $I_6$  fiber supporting the root type  $R = A_5 + 2A_2$

If the smooth rational bisection  $B$  does not produce a two-torsion section, the picture looks as follows (with a dashed line indicating that the respective fiber could also be ramified):

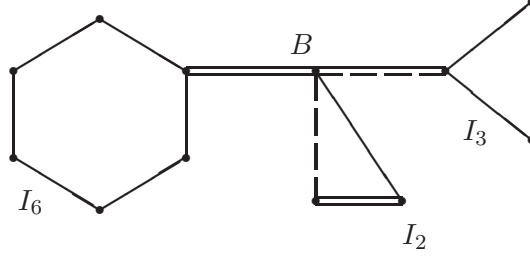


FIGURE 2. Configurations supporting the root type  $R = A_5 + 2A_2$  without multiple  $I_6$  fiber

## 6. QUADRATIC TWIST

We shall now extract conditions from the data developed in and from Table 1 and start to convert them into explicit equations towards the proof of Theorem 1.1. Continuing from Summary 4.1, we proceed as follows:

- use Lemma 4.2 to single out the possible fiber configurations supporting a given rank 9 root type  $R$  (as in Example 5.3);
- prove that there are reflections in fiber components mapping all curves form  $R_0$  to fiber components (and not affecting the smooth rational bisection  $B$ ); this amounts to a case-by-case analysis paralleling [17, §7] using standard properties of root lattices and Weyl groups (cf. the proof of Lemma 4.2 for the argument for the root type  $R = A_3 + 6A_1$ ).

Especially the second step allows us to predict exactly how the bisection  $B$  intersects the singular fibers (depending on their multiplicity, so there may be a few cases to distinguish as in Example 5.3). Note that this directly carries over to information on the section  $P$  on the K3 cover  $X$ . In particular, we can determine whether  $P$  is two-torsion or not. We now turn to the problem of exhibiting explicit equations for the Enriques surfaces in question.

**6.1. Two-torsion case.** If the configuration determines  $P$  as a two-torsion section, then there is little left to do: there always is a ramified reducible fiber (necessarily of multiplicative type), so  $X$  arises from  $\text{Jac}(f)$  by a quadratic base change ramified at this fiber. Hence such K3 surfaces (and their quotient singular Enriques surfaces) occur in one-dimensional families (depending on the free parameter of the base change).

*Example 6.1* ( $R = A_5 + 2A_2$  cont'd). In the two-torsion case from Example 5.3, there is ramification at the  $I_6$  fiber, so the base changes can be normalized to take the shape

$$t \mapsto \lambda t^2 / (t - 1) \quad (\lambda \in K^\times).$$

By inspection of Figure 1, the resulting singular Enriques surfaces also support the root types  $E_7 + A_2$  and  $A_7 + A_2$ .

**Summary 6.2.** We obtain three one-dimensional families of singular Enriques surfaces supporting the following root types:

$R$	$\text{Jac}(f)$	mult. fiber
$E_8 + A_1, D_8 + A_1, A_8 + A_1, A_7 + 2A_1$	$X_{321}$	$I_2$
$E_7 + A_2, A_7 + A_2, A_5 + 2A_2$	$X_{6321}$	$I_6$
$E_6 + A_2 + A_1$	$X_{6321}$	$I_2$

**6.2. Non-torsion case.** Here the section  $P$  is really anti-invariant for the induced action of the deck transformation  $\iota$  – or equivalently, invariant for the composition  $j = \iota \circ (-\text{id})$  where  $(-\text{id})$  denotes the involution of the generic fiber. Hence  $P$  descends to a section  $P'$  on the minimal resolution of the quotient surface,

$$X' = \widetilde{X/j}.$$

This is again an elliptic K3 surface with the same fibers at  $\text{Jac}(f)$  except at the ramified fiber (which generically is of Kodaira type  $I_4^*$ , the additional components accounting for the isolated fixed points of  $j$  in the ramified fiber, cf. the proof of Corollary 5.2).  $X'$  is often called *quadratic twist* of  $\text{Jac}(f)$ . In order to compute explicit equations, we arrange for the quadratic base change to be ramified at  $\infty$  and thus take the shape

$$(4) \quad t \mapsto t(t+1).$$

To this end, we fix a suitable (reducible) fiber at  $t = 0$  and move around the other fibers. Starting from the models in Table 2, this can be achieved by way of the Möbius transformation

$$(5) \quad t \mapsto \frac{t}{\mu t + \lambda} \quad (\lambda \neq 0),$$

for instance, in agreement with the general fact that quadratic base changes of a given rational elliptic surface come in two-dimensional families. So let us assume that the rational elliptic surface  $\text{Jac}(f)$  has Weierstrass equation

$$\text{Jac}(f) : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in K[t], \quad \deg(a_i) \leq i,$$

with singular fiber moving around according to the Möbius transformation (5). If  $X$  arises from  $\text{Jac}(f)$  by the quadratic base change (4), then one can directly compute the invariants for the involution  $j \in \text{Aut}(X)$ . They result in the following Weierstrass equation of the quadratic twist  $X'$ :

$$(6) \quad X' : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 + t(a_1x + a_3)^2.$$

Enter the section  $P \in \text{MW}(X)$  which ought be disjoint from  $O$ ; this property is often called integral and equivalent to  $P = (U, V)$  for polynomials  $U, V \in K[t]$  of degree at most 4 resp. 6. For  $P'$ , this translates as  $P' = (U', V')$  of degree at most 2 resp. 3. There is an important additional condition for the top degree coefficient of  $U'$ :

**Observation 6.3.** Write out the top degree coefficients

$$U' = u_2t^2 + \dots, \quad a_1 = a_{11}t + a_{10}, \quad a_3 = a_{33}t^3 + \dots,$$

depending on  $\mu, \lambda$ . Then  $u_2 = a_{33}/a_{11}$ .

*Proof.* With the above degree conditions on  $U', V'$ , it is clear that  $a_{11}u_2$  and  $a_{33}$  are the only degree 7 coefficients of the equation obtained from (6) upon substituting  $P'$ . Hence solving for the equation to vanish identically leads to the above relation.  $\square$

*Remark 6.4.* Observation 6.3 also implies that  $a_1 \neq 0$ .

Observation 6.3 also ensures that  $P'$  meets a far simple component of the (generic)  $I_4^*$  fiber of  $X'$  at  $\infty$  – and leads to a degree 6 polynomial equation in  $t$  as explained in the proof. Then we either try to solve directly for this equation to vanish identically or throw in some additional information, for instance about fibers met non-trivially by  $P'$  (inferred from the intersection pattern of the smooth rational bisection  $B$  with the singular fibers on the Enriques surface  $S$ , or from torsion sections present on  $X$  as we shall exploit in the next sections).

## 7. EXPLICIT EQUATIONS FOR ROOT TYPE $R = A_5 + 2A_2$

We illustrate the approach in the non-torsion case as sketched in 6.2 by elaborating on our usual exemplary root type  $R = A_5 + 2A_2$ . Consider an Enriques surface  $S$  with elliptic fibration (1) induced by the first isotropic vector in Example 2.1. Recall that  $\text{Jac}(f) = X_{6321}$  with the possible fiber configurations from Example 5.3. Since we have already settled the case of a multiple  $I_6$  fiber in Example 6.1, we shall assume that the  $I_6$  fiber is unramified. Thus it is safe to locate it at  $t = 0$  as in Table 2.

For starters, we restrict to the case where the fiber of type  $I_2$  is unramified ( $\mu \neq 1$  in (5)). Presently, we could set out to calculate the section  $P'$  directly on the quadratic twist  $X'$ , using the condition that it meets the  $I_2$ -fiber non-trivially, but it turns out to be even more beneficial to work on the K3 cover  $X$  itself and just remember that  $P$  is induced from  $X'$ . In particular, this implies that  $U$  is invariant under the deck transformation

$$\iota : t \mapsto t + 1$$

of (4) while  $V$  decidedly is not, for otherwise  $P$  would be induced from  $\text{Jac}(f)$ .

We shall facilitate that  $X$  admits a 3-torsion section at  $Q = (0, 0)$ , precisely it takes the general shape

$$(7) \quad X : y^2 + a_1xy + a_3y = x^3$$

(with reducible fibers at the zeroes of  $a_3$ , presently of Kodaira types  $I_6$  and  $IV$ ). It follows from divisibility considerations in  $K[t]$ , or more general from the theory of Mordell–Weil lattices [20], that  $P \cdot Q = 2$ . By the given shape of  $Q$ , this implies that  $U$  and  $V$  share a common factor  $g$  of degree two. Since  $P$  meets the fibers of type  $I_6$  and  $IV$  at the identity component,  $g$  is relatively prime to  $a_3$ , and we infer from vanishing orders that in fact  $g^3 \mid V$ . Recalling that  $V$  is not invariant under  $\iota$ , we deduce that

$$g(t) \neq g(t + 1) = h(t).$$

On the other hand, the invariance of  $U$  under  $\iota$  leads, after absorbing some factor into  $g$  if necessary, to

$$U = gh, \quad V = \nu g^3 \quad (\nu \in K^\times).$$

But then comparing the substitution into (7) with the relation obtained from  $\iota^*P = -P$ , we read off  $\nu = 1$ . We write out  $g = g_2t^2 + g_1t + g_0$  (with  $g_2 + g_1 \neq 0$ , for otherwise  $g = h$ ) and solve for the substitution into (7),

$$(8) \quad g^3 + a_1gh + a_3 = h^3$$

to vanish identically, taking into account additionally that the  $I_2$  fibers are met non-trivially. The top coefficients of (8) directly give  $g_2 = 1$  and  $g_1 = 1/\mu$ . Then the  $I_2$  fiber condition implies  $\lambda = g_0\mu(g_0\mu + \mu + 1)$  whence

$$g_0^2\mu^3 + g_0\mu^2(\mu + 1) + (\mu + 1)^2 = 0.$$

This rational curve is parametrized by

$$g_0 = u(\mu + 1)\mu, \quad \mu = 1/(u^2 + u).$$

All in all, this gives rise to a one-dimensional family of singular Enriques surfaces (also displayed in Table 3), supporting the root type  $A_5 + 2A_2$  as in Figure 2.

If the  $I_2$  fiber were to ramify (i.e.  $\mu = 1$ ), then the same approach would go through, locating the fiber at  $\infty$  and yielding  $g_2 = g_1 = 1$  and thus  $g = h$ , contradiction. This completes the analysis of the singular Enriques surfaces supporting the root type  $R = A_5 + 2A_2$ .

*Remark 7.1.* A similar analysis can be carried out starting from the second isotropic vector from Example 2.1. In fact, the corresponding systems of equations are straight forward to solve, thanks to the fibers met non-trivially. In case  $\text{Jac}(f) = X_{431}$ , we directly obtain

$$U' = \mu t^2 \quad \text{and} \quad V' = \lambda t^3 \quad (\text{up to exchanging } P' \text{ and } -P')$$

which subsequently leads to a one-dimensional family parametrized by  $\lambda = \sqrt{\mu}^3$ . For  $\text{Jac}(f) = X_{3333}$ , however, the resulting equations are a little more complicated to display.

In either case, the resulting families of Enriques surfaces are easily related. This can be achieved by singling out the multiple  $I_6$  fiber in the diagram of  $(-2)$ -curves underlying the configuration from Example 5.3 (or a  $IV^*$  fiber with smooth rational bisection and disjoint  $A_2$  in Figure 1), or as part of a more general pattern explored in the context of Enriques surfaces with four cusps in [16] (over  $\mathbb{C}$ , but the arguments carry over to characteristic two).

## 8. ONE-DIMENSIONAL FAMILIES

Having treated the exemplary root type  $R = A_5 + 2A_2$  in full detail, we shall now state the main classification result needed to prove Theorem 1.1.

**Theorem 8.1.** *Let  $S$  be a singular Enriques surface in characteristic two supporting a root lattice  $R$  of rank 9. Consider the section  $P$  on the  $K3$ -cover of  $S$  obtained from the data in Table 1. If  $P$  is not two-torsion, then  $S$  and  $R$  appear in Table 3.*

Table 3 involves the elliptic fibration (1) induced by the isotropic vector from Table 1 (or its Jacobian), and the  $x$ -coordinate  $U'$  of the section  $P' \in \text{MW}(X')$  for the quadratic twist  $X'$  which in turn is determined by the parameters  $\mu, \lambda$  entering in (5). Of course, we always have to exclude a few values for  $(\mu, \lambda)$  where the Enriques surfaces degenerate, but we omit the details for brevity.

By default, we usually start from the Weierstrass form of  $\text{Jac}(f)$  given in Table 1 though in one instance, the equations look much nicer when starting from the other affine standard coordinate  $s = 1/t$  of  $\mathbb{P}^1$  (to which we then apply (4) and (5) analogously).

root type	$\text{Jac}(f)$	section: $U'$	quadr. twist: $(\mu, \lambda)$
$A_9$	$X_{8211}$	$t(t + 1/\mu^2)$	$(\mu, 1/\mu)$
$A_8 + A_1$	$X_{6321}$	$t(t + (\mu + 1)/\mu^2)$	$(\mu, (\mu + 1)/\mu)$
$A_6 + A_2 + A_1$	$X_{6321}$	$t(t + (\mu + 1)/\mu^2)$	$(\mu, 1/(\mu(\mu + 1)))$
$A_5 + A_4$	$X_{6321}$	$\mu^2 s(s + \mu^2 + \mu)$	$(\mu, \mu^3)$
$A_5 + A_3 + A_1$	$X_{141}$	as $E_6 + A_3$	
$A_5 + 2A_2$	$X_{6321}$	$t^2 + u(u + 1)(u^2 + u + 1)t + u(u + 1)(u^2 + u + 1)^2$	$(1/(u^2 + u), (u^2 + u + 1)^2/(u^2 + u))$
$A_5 + A_2 + 2A_1$	$X_{6321}$	$t^2 + u(u + 1)t + u(u + 1)^2$	$(1/u, u/(1 + u)^2)$
$2A_4 + A_1$	$X_{5511}$	$(u^2 + u + 1)^2 t^2 + (u^4 + u + 1)(u + 1)^2 u^2 t + (u^4 + u + 1)^2 (u + 1)^2 u^2$	$\left( \frac{(u^2 + u + 1)^2}{(u + 1)^2 u^2}, \frac{(u^4 + u + 1)^3}{(u + 1)^4 u^4} \right)$
$3A_3$	$X_{9111}$	as $D_9$	
$A_3 + 3A_2$	$X_{3333}$	$((\lambda^6 + 1)t^2 + \lambda^3 t + \lambda^6)/\lambda^4$	$(1/\lambda^2, \lambda)$
$D_9$	$X_{9111}$	$\lambda^2 t^2$	$(1/\lambda^2, \lambda)$
$D_6 + A_3$	$X_{222}$	as $D_9$	
$D_5 + A_4$	$X_{5511}$	$u^2 t^2/(u + 1)^2$	$(u^2, u^5/(u + 1))$
$E_6 + A_3$	$X_{141}$	$\mu^2 t$	$(\mu, \mu^2)$

TABLE 3. Singular Enriques surfaces for 14 maximal root types

**8.1. Proof of Theorem 8.1.** Since we have treated one case in full detail and all others follow the same line of argument, we omit the details of the proof of Theorem 8.1 for space reasons (except for the case missing from the proof of Proposition 4.3, to be covered below in 8.2).

*Remark 8.2.* Some moduli components from characteristic zero cease to exist in characteristic two, even though the root type itself may still be supported on some singular Enriques surfaces. Those components are ruled out by the direct calculations, or alternatively, by more structural arguments as those involved in the proof of Proposition 4.3. We illustrate this by the following example.

*Example 8.3* ( $R = D_8 + A_1$ ). For the root type  $R = D_8 + A_1$ , the isotropic vector from Table 1 leads to  $R_0 = A_7 + A_1$ , embedding into the fiber configurations

$$\tilde{A}_7 + \tilde{A}_1, \quad \tilde{E}_7 + \tilde{A}_1, \quad \tilde{E}_8.$$

While the last configuration is not compatible with a smooth rational bisection meeting some simple fiber component with multiplicity one, the first configuration can only be ruled out by direct computation. Alternatively, consider the isotropic vector  $E' = d_2^\vee + h$ . This has  $R'_0 = D_6 + 2A_1$ . Using Lemma 3.2 and the above argument for  $\tilde{E}_8$ , one immediately derives the fiber configuration  $\tilde{E}_7 + \tilde{A}_1$ , and then a height argument shows that there is a ramified  $I_2$  fiber and a two-torsion section involved, i.e. we are in the first family of Summary 6.2.

*Remark 8.4.* Some computer algebra systems experience surprising difficulties in characteristic two calculations (notably factorization of polynomials, but also basic simplifications); fortunately, the present problem always provides a sanity check when verifying that the computed section  $P'$  indeed lies on the quadratic twist  $X'$ .

*Remark 8.5.* Similar ideas can be applied to study singular Enriques surfaces with finite automorphism group (see recent work of Martin [13]).

**8.2. Proof of Proposition 4.3 – ruling out  $R = 4A_2 + A_1$ .** Recall that the isotropic vector from Table 1 for root type  $R = 4A_2 + A_1$  implies  $\text{Jac}(f) = X_{3333}$  with full 3-torsion. Assuming that all reducible fibers are unramified and the K3 cover  $X$  admits an integral section  $P$  meeting all fibers in the zero component, we can proceed exactly as in Section 7 around (7). Then solving for (8) to hold, we directly obtain

$$g_2 = \frac{\sqrt{\mu}^3 + 1}{\sqrt{\mu}} \quad \text{and} \quad \lambda = \lambda(\sqrt{\mu}, g_1).$$

This sees the constant term in (8) factorize into four factors. Each can be seen to lead to degenerate cases where either  $\mu = 0$  or  $P$  meets some  $I_3$  fiber non-trivially.

Similarly, if one of the  $I_3$  fibers ramifies, then we can locate it at  $\infty$  by working with the affine parameter  $s = 1/t$  for  $X_{3333}$ . It then suffices to consider scalings ( $s \mapsto \mu s$ ) before the quadratic base change ( $s \mapsto s(s+1)$ ) to the K3 cover  $X$ . Here  $P$  still takes the shape from Section 7, plus it meets the ramified fiber non-trivially; this directly implies  $g_2 = 1$ . Then solving for (8) to hold returns  $g_1 = 1$ , i.e.  $g$  is invariant under the deck transformation ( $s \mapsto s+1$ ). Hence  $P$  is induced from  $X_{3333}$  as opposed to its quadratic twist. This contradiction concludes the proof of Proposition 4.3.  $\square$

## 9. MODULI COMPONENTS

With Theorem 8.1 and Summary 6.2 at our disposal, the proof of Theorem 1.1 is almost complete. It remains to prove statement (ii) about the moduli



components for root types

$$R = A_8 + A_1, \quad A_5 + 2A_2.$$

This amounts to verifying that the families exhibited in Table 3 resp. Summary 6.2 are indeed distinct. We shall prove the following slightly stronger statement:

**Lemma 9.1.** *Let  $R = A_8 + A_1$  or  $A_5 + 2A_2$ . There are two distinct families of K3 covers of singular Enriques surfaces supporting  $R$ .*

*Proof.* Compared to what had to be done in [18], especially in characteristics 3, the arguments are greatly simplified thanks to the fact that a singular Enriques surface in characteristic two cannot have a supersingular K3 surface as universal cover; that is,

$$\rho(X) \leq 20$$

(and, in fact, the height is one by [7, Cor. A.2]) In turn, this implies that a generic member  $X_\eta$  of either of the one-dimensional families of K3 surfaces at hand has

$$\rho(X_\eta) = 19$$

(since it is provided with plenty of smooth rational curves from the Enriques surface). We continue by comparing the discriminants  $d$  of  $\text{NS}(X_\eta)$  using [19, (11.22)]. For  $R = A_5 + 2A_2$ , this returns once  $d = 12$  and the other time  $d = 108$ . To see this, it suffices to check that there cannot be further torsion sections than the present cyclic group  $\mathbb{Z}/6\mathbb{Z}$ , and that for the second family, the section  $P$  mapping to the smooth rational bisection  $B \subset Y$  generically has height  $h(P) = 3$ . At the same time, neither  $P$  nor its translates by torsion sections may be 2-divisible since then the height would result in  $3/4$  which is not compatible with the contraction terms in the height pairing. Similarly, if  $P$  were 3-divisible, i.e.  $P = 3Q$  then  $h(Q) = 1/3$  could a priori be accommodated by certain intersection patterns, but each case would lead to a contradiction by calculating that the height pairing with some two- or six-torsion section would be negative (as opposed to being zero). Finally, translating  $P$  by some torsion section does not make a difference as the two-torsion section itself is 3-divisible and the sections of higher torsion order cause the  $I_3$  fibers to be met non-trivially, so that there cannot be any 3-divisibility at all.

The argument for  $R = A_8 + A_1$  is similar, so we leave the details to the reader.  $\square$

**Conclusion.** Lemma 9.1 proves part (ii) of Theorem 1.1. Together with Theorem 8.1 and Summary 6.2, the proof of Theorem 1.1 is thus complete.  $\square$

**Acknowledgements.** Thanks to Christian Liedtke and Toshiyuki Katsura for helpful comments.

## REFERENCES

- [1] Bombieri, E., Mumford, D.: *Enriques' classification of surfaces in char. p. II*, in: Baily, W. L. Jr., Shioda, T. (eds.), *Complex analysis and algebraic geometry*, Iwanami Shoten, Tokyo (1977) 23–42.
- [2] Bombieri, E., Mumford, D.: *Enriques' classification of surfaces in char. p. III*, Invent. Math. **35** (1976), 197–232.
- [3] Deligne, P.: *Relevement de surfaces K3 en caractéristique nulle*, Lect. Notes Math. **868** (1981), 58–71.
- [4] Hulek, K., Schütt, M.: *Enriques surfaces and Jacobian elliptic surfaces*, Math. Z. **268** (2011), 1025–1056.
- [5] Hwang, D., Keum, JH, Ohashi, H.: *Gorenstein  $\mathbb{Q}$ -homology projective planes*, Science China Math. **58** (2015), 501–512.
- [6] Illusie, L.: *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Sci. École Norm. Sup. **12** (1979), 501–661.
- [7] Katsura, T., Kondō, S.: *On Enriques surfaces in characteristic 2 with a finite group of automorphisms*, to appear in J. Alg. Geom., preprint (2015), arXiv:1512.06923.
- [8] Kondō, S.: *Enriques surfaces with finite automorphism groups.*, Japan. J. Math. (N.S.) **12** (1986), 191–282.
- [9] Lang, W. E.: *Extremal rational elliptic surfaces in characteristic p. I. Beauville surfaces*, Math. Z. **207** (1991), 429–437.
- [10] Lang, W. E.: *Extremal rational elliptic surfaces in characteristic p. II. Surfaces with three or fewer singular fibres*, Ark. Mat. **32** (1994), 423–448.
- [11] Lieblich, M., Maulik, D.: *A note on the cone conjecture for K3 surfaces in positive characteristic*, preprint (2011), arXiv: 1102.3377v3.
- [12] Liedtke, C.: *Arithmetic Moduli and Lifting of Enriques Surfaces*, J. Reine Angew. Math. **706** (2015), 35–65.
- [13] Martin, G.: *Enriques surfaces with finite automorphism group in positive characteristic*, preprint (2017), arXiv: 1703.08419.
- [14] Miranda, R., Persson, U.: *On Extremal Rational Elliptic Surfaces*, Math. Z. **193** (1986), 537–558.
- [15] Mumford, D.: *Enriques' classification of surfaces in char. p. I*, in: Global analysis. Princeton, University Press 1969.
- [16] Rams, S., Schütt, M.: *On Enriques surfaces with four cusps*, preprint (2015), arXiv: 1404.3924v2.
- [17] Schütt, M.: *Moduli of Gorenstein  $\mathbb{Q}$ -homology projective planes*, preprint (2016), arXiv: 1505.04163v2.
- [18] Schütt, M.:  *$\mathbb{Q}_\ell$ -cohomology projective planes from Enriques surfaces in odd characteristic*, preprint (2016), arXiv: 1611.03847.
- [19] Schütt, M., Shioda, T.: *Elliptic surfaces*, Algebraic geometry in East Asia - Seoul 2008, Advanced Studies in Pure Math. **60** (2010), 51–160.
- [20] Shioda, T.: *On the Mordell–Weil lattices*, Comm. Math. Univ. St. Pauli **39** (1990), 211–240.

INSTITUT FÜR ALGEBRAISCHE GEOMETRIE, LEIBNIZ UNIVERSITÄT HANNOVER, WELFENGARTEN 1, 30167 HANNOVER, GERMANY

RIEMANN CENTER FOR GEOMETRY AND PHYSICS, LEIBNIZ UNIVERSITÄT HANNOVER, APPELSTRASSE 2, 30167 HANNOVER, GERMANY

*E-mail address:* schuett@math.uni-hannover.de